Parametric solution to IBP relations

Tobias Huber
Universität Siegen

TRR 257 - Particle Physics Phenomenology after the Higgs Discovery

M. Barakat, R. Brüser, J. Piclum, TH in preparation

Loops and Legs, Ettal, April 27th, 2022

Outline

- Motivation
- Mathematical stuff
- Gröbner basis of the ideal of IBP relations in the double-shift algebra
- One-loop massless box
- Two-loop on-shell kite
- Conclusion and outlook

Motivation

- Integral reduction is an indispensable tool for higher-order calculations
- Based on integration-by-parts (IBP) relations [Tkachov'81; Chetyrkin,Tkachov'81]
- Many sophisticated public and private tools exist
  - AIR [Anastasiou,Lazopoulos'04]
  - Reduze [Studerus'09; Manteuffel,Studerus'12]
  - FIRE [Smirnov et al.'08+]
  - LiteRed [Lee'12+]
  - Kira [Maierhöfer,Usovitsch,Uwer'17; Klappert,Lange,Maierhöfer,Usovitsch'20]
  - Crusher . . . [Marquard,Seidel]

Motivation

- Integral reduction is an indispensable tool for higher-order calculations
- Based on integration-by-parts (IBP) relations [Tkachov'81; Chetyrkin,Tkachov'81]
- Many sophisticated public and private tools exist
  - AIR [Anastasiou,Lazopoulos'04]
  - Reduze [Studerus'09; Manteuffel,Studerus'12]
  - FIRE [Smirnov et al.'08+]
  - LiteRed [Lee'12+]
  - Kira [Maierhöfer,Usovitsch,Uwer'17; Klappert,Lange,Maierhöfer,Usovitsch'20]
  - Crusher ... [Marquard,Seidel]

- Mostly based on Laporta's algorithm [Laporta'01]
  - Solves IBP equations for numerical values of indices with Gaussian elimination.
- Several refinements exist, e.g.
  - Parallelization
  - Methods from finite fields [v. Manteuffel,Schabinger'14; Smirnov,Chukharev'19] [Peraro'16'19; Klappert,Klein,Lange'19'20]
- Drawbacks
  - Compute many more integrals than required
  - Large storage required for results of $10^{\sim 4-6}$ integrals

Motivation

- New ideas from
  - syzygy equations [Kosower et al.'10'18; Schabinger et al.'11'20; Ita'15; Böhm et al.'17]
  - algebraic geometry [Larsen,Zhang'14; Böhm et al.'18'19]
  - intersection numbers [Mastrolia,Mizera'18; Frellesvig et al.'19'20; Weinzierl'20]
- Our approach
  - Leave propagator powers symbolic
  - Find a Gröbner basis of the left ideal of IBP relations in the double-shift algebra
  - Derive normal form from Gröbner basis

Motivation

- New ideas from
  - syzygy equations [Kosower et al.'10'18; Schabinger et al.'11'20; Ita'15; Böhm et al.'17]
  - algebraic geometry [Larsen,Zhang'14; Böhm et al.'18'19]
  - intersection numbers [Mastrolia,Mizera'18; Frellesvig et al.'19'20; Weinzierl'20]
- Our approach
  - Leave propagator powers symbolic
  - Find a Gröbner basis of the left ideal of IBP relations in the double-shift algebra
  - Derive normal form from Gröbner basis

- Preliminary work using Gröbner bases
  - in combination with PDE and dimension shift [Tarasov'98'04]
  - in combination with shift algebras [Gerdt'05 (but chosen example is scaleless and trivial)]
  - MAPLE implementation of IBP reduction with shift algebras [Gerdt,Robertz'06]
  - introduction of sector bases (s-bases) [Smirnov,Smirnov'05-'08]
  - discussion of Gröbner bases and left and right ideals (for IBPs and scaleless ints) [Lee'08]

Reminder about mathematical stuff

Let R be a ring over a field $\mathbb{K}$.

- A subset $I \subseteq R$ is an **ideal** if it forms an additive group and fulfils

$$x \in R \wedge y \in I \implies xy \in I \ \wedge \ yx \in I.$$

  - Example: Set of even integers is an ideal in the ring of integers.
  - Left and right ideal analogous

- A **monomial order** on R is a total order $>$ s.t.

$$x^\alpha > x^\beta \implies x^\gamma x^\alpha > x^\gamma x^\beta \quad \forall\, \alpha, \beta, \gamma \in \mathbb{N}^n$$

[α, β, γ are multi-indices]

  - **Lexicographic order**

$$x^\alpha >_{\text{lex}} x^\beta \iff \text{first nonzero entry of } \alpha - \beta > 0.$$

  - **Degree reverse lexicographic order**

$$x^\alpha >_{\text{drlex}} x^\beta \iff \deg x^\alpha > \deg x^\beta \text{ or } (\deg x^\alpha = \deg x^\beta \text{ and last nonzero entry of } \alpha - \beta < 0).$$

Reminder about mathematical stuff

Let R be a ring over a field $\mathbb{K}$.

- A subset $I \subseteq R$ is an **ideal** if it forms an additive group and fulfils

$$x \in R \wedge y \in I \implies xy \in I \ \wedge \ yx \in I.$$

  - Example: Set of even integers is an ideal in the ring of integers.
  - Left and right ideal analogous

- A **monomial order** on R is a total order $>$ s.t.

$$x^\alpha > x^\beta \implies x^\gamma x^\alpha > x^\gamma x^\beta \quad \forall\, \alpha, \beta, \gamma \in \mathbb{N}^n$$

[α, β, γ are multi-indices]

  - **Lexicographic order**

$$x^\alpha >_{\text{lex}} x^\beta \iff \text{first nonzero entry of } \alpha - \beta > 0.$$

  - **Degree reverse lexicographic order**

$$x^\alpha >_{\text{drlex}} x^\beta \iff \deg x^\alpha > \deg x^\beta \text{ or } (\deg x^\alpha = \deg x^\beta \text{ and last nonzero entry of } \alpha - \beta < 0).$$

- For $f \in R$, the **leading term** $\boldsymbol{L}_>(f)$ w.r.t. $>$ is the largest term in f w.r.t. $>$.
- A finite subset $G = \{f_1, \ldots, f_r\} \subset I$ is a **Gröbner basis for $\boldsymbol{I}$** if

$$\boldsymbol{L}_>(I) = \boldsymbol{L}_>(G)\,,$$

  i.e. the leading submodule of I is generated by the leading terms of the elements of G. Hence G generates I.

- One way of computing Gröbner bases is via **Buchberger's algorithm**. It is also applicable to the non-commutative case (left ideal).
- The **remainder** h of $g = \sum_{i=1}^r g_i f_i + h$ is uniquely determined by g, I, and $>$.
- Call $\mathrm{NF}_{I,>}(g) = \mathrm{NF}_G(g) = h$ the **normal form** of g mod I w.r.t. $>$.

IBPs as left ideal in the double-shift algebra

- The following computations will take place in the **non-commutative rational** double-shift algebra

$$Y := \mathbb{Q}(D, s_{ij}, m_i^2)(a_1, \ldots, a_n)\langle D_j, D_j^- \mid j = 1, \ldots, n\rangle / (D_i D_i^- = 1 = D_i^- D_i \mid i = 1, \ldots, n)$$

in the indeterminates $a_1, \ldots, a_n, D_1, \ldots, D_n, D_1^-, \ldots, D_n^-$ with relations

$$[a_i, D_j] = \delta_{ij} D_i\,, \qquad [a_i, D_j^-] = -\delta_{ij} D_i^-, \qquad D_i D_i^- = 1,$$

[no summation over repeated indices]

$$[a_i, a_j] = [D_i, D_j] = [D_i^-, D_j^-] = [D_i, D_j^-] = 0,$$

with **partial right** action

$$I(\ldots, z_i, \ldots) \bullet D_i = I(\ldots, z_i - 1, \ldots), \qquad \underbrace{I(\ldots, z_i, \ldots)}_{\text{not scaleless}} \bullet D_i^- = I(\ldots, z_i + 1, \ldots),$$

[note that $D_i \sim i^-$ and $D_i^- \sim i^+$]

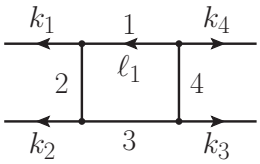
$$I(\ldots, z_i, \ldots) \bullet a_i = z_i I(\ldots, z_i, \ldots), \qquad I(\ldots, \underbrace{z_i}_{\neq 0}, \ldots) \bullet a_i^{-1} = \frac{1}{z_i} I(\ldots, z_i, \ldots).$$

- The IBP relations generate a **left** ideal $I_{\mathrm{IBP}} := \langle r_i \mid i = 1, \ldots, L(L+E)\rangle \lhd Y$.
- Goal: Compute a Gröbner basis for the left ideal I_{IBP} in Y.

Computing a Gröbner basis for $I_{\mathrm{IBP}} \lhd Y$

- The GAP package `LoopIntegrals` computes IBP relations among loop integrals. It relies on
  - the computer algebra system SINGULAR for **commutative** Gröbner bases in polynomial rings,
  - its subsystem PLURAL for non-commutative Gröbner bases in the double-shift algebra with **polynomial** coefficients,
  - Chyzak's Maple package `Ore_algebra` for noncommutative Gröbner bases in the double-shift algebra with **rational** coefficients,
  - the Julia package HECKE for simulating the reduction w.r.t. Gröbner bases in the rational double-shift algebra using linear algebra over the field of rational functions.
    HECKE uses finite-field methods to compute GCDs.

One-loop massless box



$$P_1 = -\ell_1^2,$$
$$P_2 = -(\ell_1 - k_1)^2,$$
$$P_3 = -(\ell_1 - k_1 - k_2)^2,$$
$$P_4 = -(\ell_1 + k_4)^2.$$

- Massless one-loop box, $k_i^2 = 0$
- Independent Mandelstam variables s_{12}, s_{14}
- Four standard IBPs

$$r_1 = -a_2 D_1 D_2^- - a_3 D_1 D_3^- - a_4 D_1 D_4^- - s_{12} a_3 D_3^- + (D - 2a_1 - a_2 - a_3 - a_4),$$
$$r_2 = a_1 D_1^- D_2 - a_2 D_1 D_2^- - a_3 D_1 D_3^- + a_3 D_2 D_3^- - a_4 D_1 D_4^- + a_4 D_2 D_4^- - s_{12} a_3 D_3^- + s_{14} a_4 D_4^- - a_1 + a_2,$$
$$r_3 = -a_1 D_1^- D_2 + a_1 D_1^- D_3 + a_2 D_2^- D_3 - a_3 D_2 D_3^- - a_4 D_2 D_4^- + a_4 D_3 D_4^- + s_{12} a_1 D_1^- - s_{14} a_4 D_4^- - a_2 + a_3,$$
$$r_4 = a_2 D_1 D_2^- + a_3 D_1 D_3^- - a_1 D_1^- D_4 - a_2 D_2^- D_4 - a_3 D_3^- D_4 + a_4 D_1 D_4^- - s_{14} a_2 D_2^- + s_{12} a_3 D_3^- + a_1 - a_4$$

One-loop massless box

- Reduced Gröbner basis over *rational* double-shift algebra has 9 elements

$$
\begin{aligned}
G = \Big\{ & D_4 - D_2 + \frac{(a_2 - a_4)s_{14}}{D - a_{1234}} 1, \\
& D_3 - D_1 + \frac{(a_1 - a_3)s_{12}}{D - a_{1234}} 1, \\
& \dots, \\
& 4(a_2 - 1)(D - a_{1234}) D_3 - 2(D - 2a_{134})(D - a_{1234}) D_4 + (D - 2a_{14} - 2)(D - 2a_{234}) s_{12} 1 \\
& - 2(D - 2a_{134})(a_2 - a_4) s_{14} 1 - \frac{(D - 2a_{14} - 2)(D - 2a_{34} - 2) a_4 s_{12} s_{14}}{D - a_{1234} - 1} D_4^-, \\
& - 2(D - 2a_{1234} + 4)(D - a_{1234} + 1) D_3^2 + (D - 2a_{123} + 2)(D - 2a_{134} + 2) s_{14} D_3 \\
& - 2(a_1 - a_3 + 1)(D - 2a_{1234} + 4) s_{12} D_3 + 4(a_1 - 1)(a_3 - 1) s_{12} D_4 \\
& - \frac{(D - 2a_{123} + 2)(a_3 - 1)(D - 2a_{34}) s_{12} s_{14}}{D - a_{1234}} 1 \Big\},
\end{aligned}
$$

- G is rational in D, a_i, s_{ij} and polynomial in D_i, D_i^-.

One-loop massless box

- A **standard monomial** w.r.t. the Gröbner basis G of I_{IBP} is a monomial m in the D_i, D_j^- such that $\mathrm{NF}_G(m) = m$.
- The set of standard monomials is a basis for the finite dimensional $\mathbb{K}$-vector space Y/I_{IBP}
- It corresponds to a set of master integrals with respect to e.g. the corner integral

$$\begin{aligned}
\mathrm{NF}_G(D_1) &= D_3 + \frac{(a_1 - a_3)s_{12}}{D - a_{1234}} && \rightsquigarrow D_1 \text{ is a nonstandard monomial} \\
\mathrm{NF}_G(D_2) &= D_4 + \frac{(a_2 - a_4)s_{14}}{D - a_{1234}} && \rightsquigarrow D_2 \text{ is a nonstandard monomial} \\
\mathrm{NF}_G(D_3) &= D_3 && \rightsquigarrow D_3 \text{ is a standard monomial} \\
\mathrm{NF}_G(D_4) &= D_4 && \rightsquigarrow D_4 \text{ is a standard monomial}
\end{aligned}$$

- Reveals the V_4-symmetry of the problem
- The set of standard monomials with respect to G is $\{1, D_3, D_4\}$
- Corresponds to three master integrals

$$\{I(1,1,1,1), I(1,1,0,1), I(1,1,1,0)\}.$$

- Can also verify that $D_1D_2, D_1D_4, D_2D_3, D_3D_4$ are the minimal scaleless monomials w.r.t. $I(1,1,1,1)$.

One-loop massless box

- Particularly interesting are IBP relations of the form

$$a_i D_i^- - \mathrm{NF}_G(a_i D_i^-) \in I_{\mathrm{IBP}}$$

- Normal form of operators $a_i D_i^-$ w.r.t. the Gröbner basis G of the left ideal I_{IBP}

$$\begin{aligned}
\mathrm{NF}_G(a_1 D_1^-) = &-\frac{2\,(D-2a_{124})\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{12}-2)\,(D-2a_{14}-2)\,s_{12}s_{14}}\,D_3 \\
&+\frac{4\,(a_3-1)\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{12}-2)\,(D-2a_{14}-2)\,s_{12}s_{14}}\,D_4 \\
&+\frac{(D-2a_{134})\,(D-a_{1234}-1)}{(D-2a_{14}-2)\,s_{12}}\,1, \\
\mathrm{NF}_G(a_3 D_3^-) = &-\frac{2\,(D-2a_{234})\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{23}-2)\,(D-2a_{34}-2)\,s_{12}s_{14}}\,D_3 \\
&+\frac{4\,(a_1-1)\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{23}-2)\,(D-2a_{34}-2)\,s_{12}s_{14}}\,D_4 \\
&+\frac{(D-2a_{134})\,(D-a_{1234}-1)}{(D-2a_{34}-2)\,s_{12}}\,1 \\
&-\frac{2\,(a_1-a_3)\,(D-2a_{234})\,(D-a_{1234}-1)}{(D-2a_{23}-2)\,(D-2a_{34}-2)\,s_{14}}\,1,
\end{aligned}$$

$$\begin{aligned}
\mathrm{NF}_G(a_2 D_2^-) = &\frac{4\,(a_4-1)\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{12}-2)\,(D-2a_{23}-2)\,s_{12}s_{14}}\,D_3 \\
&-\frac{2\,(D-2a_{123})\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{12}-2)\,(D-2a_{23}-2)\,s_{12}s_{14}}\,D_4 \\
&+\frac{(D-2a_{234})\,(D-a_{1234}-1)}{(D-2a_{23}-2)\,s_{14}}\,1, \\
\mathrm{NF}_G(a_4 D_4^-) = &\frac{4\,(a_2-1)\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{14}-2)\,(D-2a_{34}-2)\,s_{12}s_{14}}\,D_3 \\
&-\frac{2\,(D-2a_{134})\,(D-a_{1234})\,(D-a_{1234}-1)}{(D-2a_{14}-2)\,(D-2a_{34}-2)\,s_{12}s_{14}}\,D_4 \\
&+\frac{(D-2a_{234})\,(D-a_{1234}-1)}{(D-2a_{34}-2)\,s_{14}}\,1 \\
&-\frac{2\,(a_2-a_4)\,(D-2a_{134})\,(D-a_{1234}-1)}{(D-2a_{14}-2)\,(D-2a_{34}-2)\,s_{12}}\,1
\end{aligned}$$

- In denominators of $\mathrm{NF}_G(a_i D_i^-)$, all a_j appear together with D.
- $\mathrm{NF}_G(a_i D_i^-)$ are $\mathbb{K}$-linear combinations of the standard monomials

One-loop massless box

```
In [17]: LoadPackage( "LoopIntegrals" )

In [18]: R = RingOfLoopDiagram( LD )

Out[18]: GAP: Q[D,s12,s14][D1,D2,D3,D4]

In [19]: Ypol = DoubleShiftAlgebra( R )

Out[19]: GAP: Q[D,s12,s14][a1,a2,a3,a4]<D1,D1_,D2,D2_,D3,D3_,D4,D4_>/( D1*D1_-1, D2*D2_-1, D3*D3_-1, D4
         *D4_-1 )

In [20]: ibps = MatrixOfIBPRelations( LD )

Out[20]: GAP: <A 4 x 1 matrix over a residue class ring>

In [21]: r1 = ibps[1,1]

Out[21]: GAP: |[ -a2*D1*D2_-s12*a3*D3_-a3*D1*D3_-a4*D1*D4_+D-2*a1-a2-a3-a4 ]|

In [22]: bas_pol = BasisOfRows( ibps )

Out[22]: GAP: <A non-zero 28 x 1 matrix over a residue class ring>

In [23]: NormalForm( "a1*D1_" /  Ypol, bas_pol )

Out[23]: GAP: |[ a1*D1_ ]|
```

One-loop massless box

The following command needs Chyzak's Maple package `Ore_algebra` for the noncommutative Gröbner bases of the rational double-shift algebra:

```
In [24]: Y = RationalDoubleShiftAlgebra( R )

Out[24]: GAP: Q(D,s12,s14)(a1,a2,a3,a4)<D1,D1_,D2,D2_,D3,D3_,D4,D4_>/( D1*D1_-1, D2*D2_-1, D3*D3_-1, D4
         *D4_-1 )

In [25]: ribps = Y * ibps

Out[25]: GAP: <A 4 x 1 matrix over a residue class ring>

In [26]: bas = BasisOfRows( ribps )

Out[26]: GAP: <A non-zero 9 x 1 matrix over a residue class ring>

In [27]: NormalForm( "a1*D1_" /  Y, bas )

Out[27]: GAP: |[ -2*(D-2*a1-2*a2-2*a4)*(D-a1-a2-a3-a4)*(-a4-1+D-a1-a2-a3)*D3/(D-2*a1-2*a4-2)/(D-2*a1-2*
         a2-2)/s12/s14+4*(a3-1)*(D-a1-a2-a3-a4)*(-a4-1+D-a1-a2-a3)*D4/(D-2*a1-2*a4-2)/(D-2*a1-2*a2-2)/s
         12/s14+(-a4-1+D-a1-a2-a3)*(D-2*a1-2*a3-2*a4)/(D-2*a1-2*a4-2)/s12 ]|
```

One-loop massless box

```
In [28]: NormalFormWrtInitialIntegral( "D1_" / Y, bas )
Out[28]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [29]: NormalFormWrtInitialIntegral( "D3_" / Y, bas )
Out[29]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [30]: NormalFormWrtInitialIntegral( "D1*D2" / Y, bas )
Out[30]: GAP: |[ 0 ]|
```

One-loop massless box

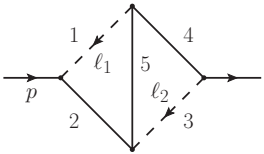
```
In [28]: NormalFormWrtInitialIntegral( "D1_" / Y, bas )
Out[28]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [29]: NormalFormWrtInitialIntegral( "D3_" / Y, bas )
Out[29]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [30]: NormalFormWrtInitialIntegral( "D1*D2" / Y, bas )
Out[30]: GAP: |[ 0 ]|
```

- Virtues of the Gröbner basis / normal form
  - Contains the entire information required for reduction
  - Conjecturally recognizes symmetries and scaleless sectors
  - No new bottom-up reduction required for new/additional integrals
  - Ideally suited for storage, e.g. in a database

Two-loop on-shell kite



$$
\begin{aligned}
P_1 &= -\ell_1^2, \\
P_2 &= m^2 - (\ell_1 + p)^2, \\
P_3 &= m^2 - (\ell_2 + p)^2, \\
P_4 &= -\ell_2^2, \\
P_5 &= m^2 - (\ell_1 + \ell_2 + p)^2
\end{aligned}
$$

- On-shell kite, $p^2 = m^2 \equiv s$

- Gröbner basis not yet available
- Simulating, using linear algebra, the computation of the normal form of $a_i D_i^-$ w.r.t. a Gröbner basis

$$
\mathrm{NF}(a_1 D_1^-) = \frac{p_{10}}{4 d_1 d_2 d_3 d_4 d_7 d_8 s} + \frac{p_{12} D_2 + p_{13} D_3 + p_{14} D_4 + p_{15} D_5}{16 d_1 d_2 d_3 d_4 d_7 d_8 d_9 s^2}, \ \ldots
$$

- p_i and d_i are expressions in the field $\mathbb{Q}(D, s_{ij}, m_i^2)(a_1, \ldots, a_n)$.
- The p_i are too large for printing, the d_i are small, e.g. $d_1 = 2a_1 + a_2 + a_3 + 2a_4 + a_5 - 2D + 1$.
- Allows for reduction of top-level sector

Conclusion

Conclusion

- We established the rational double-shift algebra and its partial action on loop integrals
- Master integrals correspond to the standard monomials w.r.t. the left ideal I_{IBP} of IBP relations
- For the one-loop massless box, the reduction problem is completely solved using the Gröbner basis of I_{IBP} in the rational double-shift algebra
- The slightly more complicated two-loop on-shell kite shows a rapid swell of the involved expressions.

Outlook / wishlist

- In general: want more loops, more legs, more scales
  - However: New ideas required to deal with enormous expression swell with increasing complexity.
- Establish a database to store results of Gröbner bases or normal forms.