Motivation

Mathematical stuff

Gröbner basis of the ideal of IBP relations in the double-shift algebra

One-loop massless box

Two-loop on-shell kite

Conclusion and outlook
Integral reduction is an indispensable tool for higher-order calculations

Based on integration-by-parts (IBP) relations

Many sophisticated public and private tools exist

- AIR [Anastasiou,Lazopoulos'04]
- Reduze [Studerus'09; Manteuffel,Studerus'12]
- FIRE [Smirnov et al.'08+]
- LiteRed [Lee’12+]
- Kira [Maierhöfer,Usovitsch,Uwer’17; Klappert,Lange, Maierhöfer,Usovitsch'20]
- Crusher . . . [Marquard,Seidel]
Motivation

- Integral reduction is an indispensable tool for higher-order calculations
- Based on integration-by-parts (IBP) relations
  \[\text{[Tkachov'81; Chetyrkin,Tkachov'81]}\]
- Many sophisticated public and private tools exist
  - AIR \[\text{[Anastasiou,Lazopoulos'04]}\]
  - Reduce \[\text{[Studerus'09; Manteuffel,Studerus'12]}\]
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  - Kira \[\text{[Maierhöfer,Usositsch,Uwer'17; Klappert,Lange,Maierhöfer,Usositsch'20]}\]
  - Crusher \[\text{[Marquard,Seidel]}\]

- Mostly based on Laporta’s algorithm \[\text{[Laporta'01]}\]
  - Solves IBP equations for numerical values of indices with Gaussian elimination.

- Several refinements exist, e.g.
  - Parallelization
  - Methods from finite fields
    \[\text{[v. Manteuffel, Schabinger'14; Smirnov, Chukharev'19]}\]
    \[\text{[Peraro'16'; Klappert, Klein, Lange'19'20]}\]

- Drawbacks
  - Compute many more integrals than required
  - Large storage required for results of \(10^{4−6}\) integrals
Motivation

- **New ideas from**
  - syzygy equations
    [Kosower et al.'10'18; Schabinger et al.'11’20; Ita’15; Böhm et al.'17]
  - algebraic geometry
    [Larsen,Zhang’14; Böhm et al.’18’19]
  - intersection numbers
    [Mastrolia,Mizera’18; Frellesvig et al.’19’20; Weinzierl’20]

- **Our approach**
  - Leave propagator powers symbolic
  - Find a Gröbner basis of the left ideal of IBP relations in the double-shift algebra
  - Derive normal form from Gröbner basis
Motivation

New ideas from

- syzygy equations [Kosower et al.'10’18; Schabinger et al.'11’20; Ita’15; Böhm et al.'17]
- algebraic geometry [Larsen, Zhang’14; Böhm et al.'18’19]
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Our approach

- Leave propagator powers symbolic
- Find a Gröbner basis of the left ideal of IBP relations in the double-shift algebra
- Derive normal form from Gröbner basis

Preliminary work using Gröbner bases

- in combination with PDE and dimension shift [Tarasov’98’04]
- in combination with shift algebras [Gerdt’05 (but chosen example is scaleless and trivial)]
- MAPLE implementation of IBP reduction with shift algebras [Gerdt, Robertz’06]
- introduction of sector bases (s-bases) [Smirnov, Smirnov’05’08]
- discussion of Gröbner bases and left and right ideals (for IBPs and scaleless ints) [Lee’08]
Let $R$ be a ring over a field $\mathbb{K}$.

- A subset $I \subseteq R$ is an **ideal** if it forms an additive group and fulfils
  \[ x \in R \land y \in I \implies xy \in I \land yx \in I. \]

- Example: Set of even integers is an ideal in the ring of integers.
- Left and right ideal analogous

- **A monomial order** on $R$ is a total order $>$ s.t.
  \[ x^\alpha > x^\beta \implies x^\gamma x^\alpha > x^\gamma x^\beta \quad \forall \alpha, \beta, \gamma \in \mathbb{N}^n \]
  
  **Lexicographic order**
  \[ x^\alpha >_{\text{lex}} x^\beta \iff \text{first nonzero entry of } \alpha - \beta > 0. \]

  **Degree reverse lexicographic order**
  \[ x^\alpha >_{\text{drlex}} x^\beta \iff \text{deg } x^\alpha > \text{deg } x^\beta \text{ or (deg } x^\alpha = \text{deg } x^\beta \text{ and last nonzero entry of } \alpha - \beta < 0). \]
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  \[ [\alpha, \beta, \gamma \text{ are multi-indices}] \]
  - **Lexicographic order**
    \[ x^\alpha >_{\text{lex}} x^\beta \iff \text{first nonzero entry of } \alpha - \beta > 0. \]
  - **Degree reverse lexicographic order**
    \[ x^\alpha >_{\text{drlex}} x^\beta \iff \deg x^\alpha > \deg x^\beta \text{ or } (\deg x^\alpha = \deg x^\beta \text{ and last nonzero entry of } \alpha - \beta < 0). \]

- For $f \in R$, the **leading term** $L_>(f)$ w.r.t. $>$ is the largest term in $f$ w.r.t. $>$.
- A finite subset $G = \{f_1, \ldots, f_r\} \subset I$ is a **Gröbner basis for $I$** if
  \[ L_>(I) = L_>(G), \]
  i.e. the leading submodule of $I$ is generated by the leading terms of the elements of $G$. Hence $G$ generates $I$.
- One way of computing Gröbner bases is via **Buchberger’s algorithm**. It is also applicable to the non-commutative case (left ideal).
- The **remainder** $h$ of $g = \sum_{i=1}^r g_i f_i + h$ is uniquely determined by $g$, $I$, and $>.$
- Call $\text{NF}_{I,>}(g) = \text{NF}_G(g) = h$ the **normal form** of $g$ mod $I$ w.r.t. $>.$
The following computations will take place in the **non-commutative rational** double-shift algebra

\[
Y := \mathbb{Q}(D, s_{ij}, m_i^2)(a_1, \ldots, a_n)\langle D_j, D_j^- | j = 1, \ldots, n \rangle / (D_i D_i^- = 1 = D_i^- D_i | i = 1, \ldots, n)
\]

in the indeterminates \(a_1, \ldots, a_n, D_1, \ldots, D_n, D_1^-, \ldots, D_n^-\) with relations

\[
\begin{align*}
[a_i, D_j] &= \delta_{ij} D_i, & [a_i, D_j^-] &= -\delta_{ij} D_i^- , & D_i D_i^- &= 1 , \\
[a_i, a_j] &= [D_i, D_j] = [D_i^-, D_j^-] = [D_i, D_j^-] = 0 ,
\end{align*}
\]

with **partial right** action

\[
I(\ldots, z_i, \ldots) \bullet D_i = I(\ldots, z_i - 1, \ldots), \quad I(\ldots, z_i, \ldots) \bullet D_i^- = I(\ldots, z_i + 1, \ldots),
\]

\[
I(\ldots, z_i, \ldots) \bullet a_i = z_i I(\ldots, z_i, \ldots), \quad I(\ldots, z_i, \ldots) \bullet a_i^{-1} = \frac{1}{z_i} I(\ldots, z_i, \ldots).
\]

The IBP relations generate a **left** ideal \(I_{\text{IBP}} := \langle r_i | i = 1, \ldots, L(L + E) \rangle \triangleleft Y\).

Goal: Compute a Gröbner basis for the left ideal \(I_{\text{IBP}}\) in \(Y\).
The GAP package LoopIntegrals computes IBP relations among loop integrals. It relies on

- the computer algebra system SINGULAR for commutative Gröbner bases in polynomial rings,
- its subsystem PLURAL for non-commutative Gröbner bases in the double-shift algebra with polynomial coefficients,
- Chyzak’s Maple package Ore_algebra for noncommutative Gröbner bases in the double-shift algebra with rational coefficients,
- the Julia package HECKE for simulating the reduction w.r.t. Gröbner bases in the rational double-shift algebra using linear algebra over the field of rational functions. HECKE uses finite-field methods to compute GCDs.
Massless one-loop box, $k_1^2 = 0$

Independent Mandelstam variables $s_{12}, s_{14}$

Four standard IBPs

\[
\begin{align*}
P_1 &= -\ell_1^2, \\
P_2 &= -(\ell_1 - k_1)^2, \\
P_3 &= -(\ell_1 - k_1 - k_2)^2, \\
P_4 &= -(\ell_1 + k_2)^2.
\end{align*}
\]

\[
\begin{align*}
r_1 &= -a_1 D_1 D_2 - a_3 D_1 D_3 - a_4 D_1 D_4 - s_{12} a_3 D_3 + (D - 2a_1 - a_2 - a_3 - a_4), \\
r_2 &= a_1 D_1 D_2 - a_2 D_1 D_2 - a_3 D_1 D_3 + a_2 D_3 - a_4 D_1 D_4 + a_4 D_2 D_4 - s_{12} a_3 D_3 + s_{14} a_4 D_4 - a_1 + a_2, \\
r_3 &= -a_1 D_1 D_2 + a_1 D_1 D_3 + a_2 D_2 D_3 - a_3 D_2 D_3 - a_4 D_2 D_4 + a_4 D_3 D_4 + s_{12} a_1 D_1 - s_{14} a_4 D_4 - a_2 + a_3, \\
r_4 &= a_2 D_1 D_2 + a_3 D_1 D_3 - a_1 D_1 D_4 - a_2 D_2 D_4 - a_3 D_3 D_4 + a_4 D_1 D_4 - s_{14} a_2 D_2 + s_{12} a_3 D_3 + a_1 - a_4
\end{align*}
\]
One-loop massless box

- Reduced Gröbner basis over *rational* double-shift algebra has 9 elements

\[
G = \left\{ D_4 - D_2 + \frac{(a_2 - a_4)s_{14}}{D - a_{1234}} \right.,
\]
\[
D_3 - D_1 + \frac{(a_1 - a_3)s_{12}}{D - a_{1234}} \right.,
\]
\[
\ldots,
\]
\[
4(a_2 - 1)(D - a_{1234})D_3 - 2(D - 2a_{134})(D - a_{1234})D_4 + (D - 2a_{14} - 2)(D - 2a_{234})s_{12}1
\]
\[
- 2(D - 2a_{134})(a_2 - a_4)s_{14}1 - \left( \frac{D - 2a_{14} - 2}{D - a_{1234} - 1} \right) (D - 2a_{34} - 2)a_4s_{12}s_{14} D_4^-, 
\]
\[
- 2(D - 2a_{1234} + 4)(D - a_{1234} + 1)D_3^2 + (D - 2a_{123} + 2)(D - 2a_{134} + 2)s_{14}D_3
\]
\[
- 2(a_1 - a_3 + 1)(D - 2a_{1234} + 4)s_{12}D_3 + 4(a_1 - 1)(a_3 - 1)s_{12}D_4
\]
\[
- \left( \frac{D - 2a_{123} + 2}{D - a_{1234}} \right)(a_3 - 1)(D - 2a_{34})s_{12}s_{14} D_4^-, 
\}\}

- \( G \) is rational in \( D, a_i, s_{ij} \) and polynomial in \( D_i, D_i^- \).
A **standard monomial** w.r.t. the Gröbner basis $G$ of $I_{IBP}$ is a monomial $m$ in the $D_i, D_j$ such that $NF_G(m) = m$.

The set of standard monomials is a basis for the finite dimensional $\mathbb{K}$-vector space $Y/I_{IBP}$.

It corresponds to a set of master integrals with respect to e.g. the corner integral:

- $NF_G(D_1) = D_3 + \frac{(a_1 - a_3)s_{12}}{D - a_{1234}} \leadsto D_1$ is a nonstandard monomial
- $NF_G(D_2) = D_4 + \frac{(a_2 - a_4)s_{14}}{D - a_{1234}} \leadsto D_2$ is a nonstandard monomial
- $NF_G(D_3) = D_3 \leadsto D_3$ is a standard monomial
- $NF_G(D_4) = D_4 \leadsto D_4$ is a standard monomial

Reveals the $V_4$-symmetry of the problem.

The set of standard monomials with respect to $G$ is $\{1, D_3, D_4\}$.

Corresponds to three master integrals

$$\{I(1, 1, 1, 1), I(1, 1, 0, 1), I(1, 1, 1, 0)\}.$$ 

Can also verify that $D_1 D_2, D_1 D_4, D_2 D_3, D_3 D_4$ are the minimal scaleless monomials w.r.t. $I(1, 1, 1, 1)$. 

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**References**

Particularly interesting are IBP relations of the form

$$a_i D_i^\sim - NF_G(a_i D_i^\sim) \in I_{IBP}$$

Normal form of operators $a_i D_i^\sim$ w.r.t. the Gröbner basis $G$ of the left ideal $I_{IBP}$

$$NF_G(a_1 D_1^\sim) = -\frac{2 (D - 2 a_{124}) (D - a_{1234}) (D - a_{1234} - 1)}{(D - 2 a_{12} - 2) (D - 2 a_{14} - 2) s_{12} s_{14}} D_3$$

$$+ \frac{4 (a_3 - 1) (D - a_{1234}) (D - a_{1234} - 1)}{(D - 2 a_{12} - 2) (D - 2 a_{14} - 2) s_{12} s_{14}} D_4$$

$$+ \frac{(D - 2 a_{134}) (D - a_{1234} - 1)}{(D - 2 a_{14} - 2) s_{12}} 1,$$

$$NF_G(a_3 D_3^\sim) = -\frac{2 (D - 2 a_{234}) (D - a_{1234}) (D - a_{1234} - 1)}{(D - 2 a_{23} - 2) (D - 2 a_{34} - 2) s_{12} s_{14}} D_3$$

$$+ \frac{4 (a_1 - 1) (D - a_{1234}) (D - a_{1234} - 1)}{(D - 2 a_{23} - 2) (D - 2 a_{34} - 2) s_{12} s_{14}} D_4$$

$$+ \frac{(D - 2 a_{134}) (D - a_{1234} - 1)}{(D - 2 a_{34} - 2) s_{12}} 1,$$

$$- \frac{2 (a_1 - a_3) (D - 2 a_{234}) (D - a_{1234} - 1)}{(D - 2 a_{23} - 2) (D - 2 a_{34} - 2) s_{14}} 1,$$

$$NF_G(a_4 D_4^\sim) = -\frac{4 (a_2 - 1) (D - a_{1234}) (D - a_{1234} - 1)}{(D - 2 a_{14} - 2) (D - 2 a_{34} - 2) s_{12} s_{14}}$$

$$+ \frac{2 (D - 2 a_{134}) (D - a_{1234}) (D - a_{1234} - 1)}{(D - 2 a_{14} - 2) (D - 2 a_{34} - 2) s_{12} s_{14}} D_4$$

$$+ \frac{(D - 2 a_{234}) (D - a_{1234} - 1)}{(D - 2 a_{34} - 2) s_{14}} 1,$$

$$- \frac{2 (a_2 - a_4) (D - 2 a_{134}) (D - a_{1234} - 1)}{(D - 2 a_{14} - 2) (D - 2 a_{34} - 2) s_{12}} 1.$$
In [17]: LoadPackage( "LoopIntegrals" )

In [18]: R = RingOfLoopDiagram( LD )

Out[18]: GAP: Q[D,s12,s14][D1,D2,D3,D4]

In [19]: Ypol = DoubleShiftAlgebra( R )

Out[19]: GAP: Q[D,s12,s14][a1,a2,a3,a4]<D1,D1_,D2,D2_,D3,D3_,D4,D4_>/ ( D1*D1_-1, D2*D2_-1, D3*D3_-1, D4 *D4_-1 )

In [20]: ibps = MatrixOfIBPRelations( LD )

Out[20]: GAP: <A 4 x 1 matrix over a residue class ring>

In [21]: r1 = ibps[1,1]

Out[21]: GAP: | [ -a2*D1*D2_-s12*a3*D3_-a3*D1*D3_-a4*D1*D4_+D-2*a1-a2-a3-a4 ] |

In [22]: bas_pol = BasisOfRows( ibps )

Out[22]: GAP: <A non-zero 28 x 1 matrix over a residue class ring>

In [23]: NormalForm( "a1*D1_" / Ypol, bas_pol )

Out[23]: GAP: | [ a1*D1_ ] |
The following command needs Chyzak's Maple package Ore_algebra for the noncommutative Gröbner bases of the rational double-shift algebra:

\begin{verbatim}
In [24]: Y = RationalDoubleShiftAlgebra( R )
Out[24]: GAP: Q(D,s12,s14)(a1,a2,a3,a4)<D1,D1_,D2,D2_,D3,D3_,D4,D4_>/ ( D1*D1_-1, D2*D2_-1, D3*D3_-1, D4*D4_-1 )
In [25]: ribps = Y * ibps
Out[25]: GAP: <A 4 x 1 matrix over a residue class ring>
In [26]: bas = BasisOfRows( ribps )
Out[26]: GAP: <A non-zero 9 x 1 matrix over a residue class ring>
In [27]: NormalForm( "a1*D1_" / Y, bas )
Out[27]: GAP: || [-2*(D-2*a1-2*a2-2*a4)*(D-a1-a2-a3-a4)*(-a4-1+D-a1-a2-a3)*D3/(D-2*a1-2*a4-2)/(D-2*a1-2*a2-2)/s12/s14+4*(a3-1)*(D-a1-a2-a3-a4)*(-a4-1+D-a1-a2-a3)*D4/(D-2*a1-2*a4-2)/(D-2*a1-2*a2-2)/s12/s14+(-a4-1+D-a1-a2-a3)*(D-2*a1-2*a3-2*a4)/(D-2*a1-2*a4-2)/s12 ] ||
\end{verbatim}
### One-loop massless box

The Virtues of the Gröbner basis / normal form contain the entire information required for reduction. Conjecturally, it recognizes symmetries and scaleless sectors. No new bottom-up reduction is required for new/additional integrals. It is ideally suited for storage, e.g., in a database.

```plaintext
In [28]: NormalFormWrtInitialIntegral( "D1_" / Y, bas )
Out[28]: GAP: [[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]]

In [29]: NormalFormWrtInitialIntegral( "D3_" / Y, bas )
Out[29]: GAP: [[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]]

In [30]: NormalFormWrtInitialIntegral( "D1*D2" / Y, bas )
Out[30]: GAP: [[ 0 ]]```
Virtues of the Gröbner basis / normal form

- Contains the entire information required for reduction
- Conjecturally recognizes symmetries and scaleless sectors
- No new bottom-up reduction required for new/additional integrals
- Ideally suited for storage, e.g. in a database
Two-loop on-shell kite

\[ P_1 = -\ell_1^2, \]
\[ P_2 = m^2 - (\ell_1 + p)^2, \]
\[ P_3 = m^2 - (\ell_2 + p)^2, \]
\[ P_4 = -\ell_2^2, \]
\[ P_5 = m^2 - (\ell_1 + \ell_2 + p)^2 \]

- On-shell kite, \( p^2 = m^2 \equiv s \)
- Gröbner basis not yet available
- Simulating, using linear algebra, the computation of the normal form of \( a_i D^-_i \) w.r.t. a Gröbner basis

\[
\text{NF}(a_1 D^-_1) = \frac{p_{10}}{4d_1 d_2 d_3 d_4 d_7 d_8 s} + \frac{p_{12} D_2 + p_{13} D_3 + p_{14} D_4 + p_{15} D_5}{16d_1 d_2 d_3 d_4 d_7 d_8 d_9 s^2}, \ldots
\]

- \( p_i \) and \( d_i \) are expressions in the field \( \mathbb{Q}(D, s_{ij}, m_i^2)(a_1, \ldots, a_n)\).
- The \( p_i \) are too large for printing, the \( d_i \) are small, e.g. \( d_1 = 2a_1 + a_2 + a_3 + 2a_4 + a_5 - 2D + 1 \).
- Allows for reduction of top-level sector
We established the rational double-shift algebra and its partial action on loop integrals.

Master integrals correspond to the standard monomials w.r.t. the left ideal $I_{\text{IBP}}$ of IBP relations.

For the one-loop massless box, the reduction problem is completely solved using the Gröbner basis of $I_{\text{IBP}}$ in the rational double-shift algebra.

The slightly more complicated two-loop on-shell kite shows a rapid swell of the involved expressions.

Outlook / wishlist

In general: want more loops, more legs, more scales.

However: New ideas required to deal with enormous expression swell with increasing complexity.

Establish a database to store results of Gröbner bases or normal forms.